

RANDOM CURVES BY CONFORMAL WELDING

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ABSTRACT. We construct a conformally invariant random family of closed curves in the plane by welding of random homeomorphisms of the unit circle given in terms of the exponential of Gaussian Free Field. We conjecture that our curves are locally related to SLE(κ) for $\kappa < 4$.

1. INTRODUCTION

A major breakthrough in the study of conformally invariant random curves in the plane occurred when O. Schramm [14] introduced the Schramm-Loewner Evolution (SLE), a stochastic process which describes such curves growing in a fictitious time so that the curve of interest is obtained as time tends to infinity. In this note we summarize a different construction [3] of random curves which is stationary i.e. the probability measure on curves is directly defined without introducing an auxiliary time. We carry out this construction for closed curves, a case that is not naturally covered by SLE.

Our construction is based on the idea of conformal welding which provides a correspondence between Jordan curves on the extended plane $\widehat{\mathbb{C}}$ and a set of homeomorphisms of the circle \mathbb{T} . Given a Jordan curve $\Gamma \subset \widehat{\mathbb{C}}$, let

$$f_+ : \mathbb{D} \rightarrow \Omega_+ \quad \text{and} \quad f_- : \mathbb{D}_\infty \rightarrow \Omega_-$$

be a choice of Riemann mappings of the unit disc \mathbb{D} and its complement onto the components of $\widehat{\mathbb{C}} \setminus \Gamma = \Omega_+ \cup \Omega_-$. By Caratheodory's theorem f_- and f_+ both extend continuously to $\partial\mathbb{D} = \partial\mathbb{D}_\infty$, and thus

$$(1) \quad \phi = f_+^{-1} \circ f_-$$

is a homeomorphism of \mathbb{T} . In the welding problem we are asked to invert this process; given a homeomorphism $\phi : \mathbb{T} \rightarrow \mathbb{T}$ we are to find a Jordan curve Γ and conformal mappings f_\pm onto the complementary domains Ω_\pm so that (1) holds. It is clear that the welding problem, when solvable, has natural conformal invariance attached to it; any image of the curve Γ under a Möbius transformation of $\widehat{\mathbb{C}}$ is equally a welding curve. Similarly, if $\phi : \mathbb{T} \rightarrow \mathbb{T}$ admits a welding, then so do all its compositions with Möbius transformations of the disk.

We solve the welding problem for a random, locally scale invariant set of homeomorphisms $h_\omega : \mathbb{T} \rightarrow \mathbb{T}$, thereby obtaining a random set of Jordan curves. To define h , identify the circle as $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0, 1)$. Given a positive Borel measure τ without atoms we get a homeomorphism $h : [0, 1) \rightarrow [0, 1)$ by

$$(2) \quad h(t) = \tau([0, t)) / \tau([0, 1))$$

It was proposed by the second author some years ago that a natural class of homeomorphisms h is obtained by taking τ formally proportional to $e^{\beta X(t)} dt$ where $\beta \geq 0$ and X is the Gaussian Free Field on the circle i.e. the random field X with covariance

$$(3) \quad \mathbb{E} X(t)X(t') = -\log |e^{2\pi it} - e^{2\pi it'}|.$$

For a rigorous definition one introduces a regularization X_ε which is a.s. continuous if $\varepsilon > 0$ and shows that almost surely the weak limit of Borel measures

$$(4) \quad \tau(dz) = \lim_{\varepsilon \rightarrow 0} e^{\beta X_\varepsilon(z)} / \mathbb{E} e^{\beta X_\varepsilon(z)} dz$$

exists and defines a non-atomic random Borel measure on $[0, 1]$.

Our main result is then:

Theorem 1.1. *For $\beta^2 < 2$ and almost surely in ω , the formula (4) defines a Hölder continuous circle homeomorphism, such that the welding problem has a solution γ , where γ is a Jordan curve bounding a domain $\Omega = f_+(\mathbb{D})$ with a Hölder continuous Riemann mapping f_+ . For a given ω , the solution is unique up to a Möbius map of the plane.*

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The "critical (inverse) temperature" $\beta_c = \sqrt{2}$ corresponds to loss of continuity of the maps h . For $\beta \geq \beta_c$ the limit (4) is zero almost surely. Based on theoretical physics [9] one may conjecture that a corresponding limit of the normalized measures $\tau_\varepsilon/\tau_\varepsilon([0, 1])$ is non trivial also for $\beta \geq \beta_c$ and atomic for $\beta > \beta_c$ thereby giving rise to a discontinuous map h . This phase transition is closely connected to the one observed in two dimensional Liouville Quantum Gravity [7] where a two dimensional version of our measure τ is considered.

We conjecture that the curves γ locally "look like" $\text{SLE}(2\beta^2)$ (see also [8] for arguments to this direction). The case $\beta = \beta_c$, presumably corresponding to $\text{SLE}(4)$, is not covered by our analysis.

It would also be of interest to understand the connection of our weldings to those arising from stochastic flows studied in the interesting work [1]. In [1] a program was set up for studying weldings that correspond to Hölder continuous homeomorphisms, but the boundary behaviour of the welding maps and hence the existence and uniqueness the welding was left open.

2. BELTRAMI EQUATION

A powerful way to solve the welding problem goes by using the Beltrami equation. Assume a homeomorphism $\phi : \mathbb{T} \rightarrow \mathbb{T}$ is extended to a locally quasiconformal map $f : \mathbb{D} \rightarrow \mathbb{D}$, i.e. $f \in C(\overline{\mathbb{D}})$ is a homeomorphism with ∇f locally integrable in \mathbb{D} and satisfying

$$(5) \quad \frac{\partial f}{\partial \bar{z}} = \mu(z) \frac{\partial f}{\partial z}, \quad \text{for a.e. } z \in \mathbb{D},$$

with $\sup_{z \in K} |\mu(z)| < 1$ for $K \subset\subset D$. One then considers the modified equation

$$(6) \quad \frac{\partial F}{\partial \bar{z}} = \chi_{\mathbb{D}}(z) \mu(z) \frac{\partial F}{\partial z}, \quad \text{for a.e. } z \in \mathbb{C}.$$

Suppose we can find a solution F to (6) which is a homeomorphism of $\widehat{\mathbb{C}}$. Then $\Gamma = F(\mathbb{T})$ is a Jordan curve. Moreover, as $\partial_{\bar{z}} F = 0$ for $|z| > 1$, we can set $f_- := F|_{\mathbb{D}_\infty}$ and $\Omega_- := F(\mathbb{D}_\infty)$ to define a conformal mapping

$$f_- : \mathbb{D}_\infty \rightarrow \Omega_-$$

To get the mapping f_+ note that both f and F solve the Beltrami equation (5) in the unit disk \mathbb{D} . By the uniqueness properties of equation (5) we conclude that

$$(7) \quad F(z) = f_+ \circ f(z), \quad z \in \mathbb{D},$$

for some conformal mapping $f_+ : \mathbb{D} = f(\mathbb{D}) \rightarrow \Omega_+ := F(\mathbb{D})$. Then, on the unit circle,

$$(8) \quad \phi(z) = f|_{\mathbb{T}}(z) = f_+^{-1} \circ f_-(z), \quad z \in \mathbb{T}$$

and we have found a solution to the welding problem.

To carry out this set of ideas we observe first that any homeomorphic self map ϕ of the circle can be extended to a locally quasiconformal map $f : \mathbb{D} \rightarrow \mathbb{D}$ via the Beurling-Ahlfors extension. However, a highly nontrivial problem remains: when does the auxiliary equation (6) have a locally quasiconformal solution and when is this unique up to a conformal map?

A classical case where this question can be answered positively is the uniformly elliptic one where there is an extension with $\|\mu\|_\infty < 1$. This in turn will be true if ϕ is quasisymmetric. In our case these conditions *do not* hold, and we are forced outside the uniformly elliptic PDE's and need to study (5) with strongly degenerate coefficients with only $|\mu(z)| < 1$ almost everywhere.

3. EXISTENCE: LEHTO METHOD

We use the method due to Lehto [12] to show the existence of homeomorphic solutions to (6). This approach is based on controlling the conformal moduli of images of annular regions. To recall his result, define the distortion function

$$K(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|},$$

corresponding to the complex dilatation $\mu = \mu(z)$. Given an annulus $A(w, r, R) := \{z \in \mathbb{C} : r < |z - w| < R\}$ define the *Lehto integral*:

$$(9) \quad L(w, r, R) := \int_r^R \frac{1}{\int_0^{2\pi} K(w + \rho e^{i\theta}) d\theta} \frac{d\rho}{\rho}$$

Lehto's theorem (see [2, p. 584]) states then that if μ is compactly supported with $|\mu(z)| < 1$ a.e., if $K(z)$ is locally integrable, and if for some $R_0 > 0$ the Lehto integral satisfies

$$(10) \quad L(z, 0, R_0) = \infty, \quad \text{for all } z \in \mathbb{C}$$

then the Beltrami equation (6) admits a homeomorphic $W_{loc}^{1,1}$ -solution $F : \mathbb{C} \rightarrow \mathbb{C}$.

We need actually a stronger result on the Lehto integrals to obtain Hölder continuity of the solution. The Lehto integral controls the geometric distortion of an annulus under a locally quasiconformal map. Indeed, given a bounded (topological) annulus $A \subset \mathbb{C}$, with E the bounded component of $\mathbb{C} \setminus A$, we denote by $D_O(A) := \text{diam}(A)$ the outer diameter, and by $D_I(A) := \text{diam}(E)$ the inner diameter of A . It then holds that for a quasiconformal map f

$$D_I(f(A(w, r, R))) \leq 16 \exp(-2\pi^2 L(w, r, R)) D_O(f(A(w, r, R))).$$

The equation (6) is solved by considering a regularized uniformly elliptic equation where μ is replaced by $(1 - \varepsilon)\mu$. Since the corresponding solutions F_ε are conformal in \mathbb{D}_∞ the outer radii $D_O(F_\varepsilon(A(w, r, 1)))$ for $w \in \mathbb{D}$ are uniformly bounded by Koebe. Thus an estimate

$$L(w, r, 1) \geq a \log 1/r$$

leads to $D_I(F_\varepsilon(A(w, r, 1))) \leq Cr^{2\pi^2 a}$ i.e. to Hölder continuity of F_ε uniformly in ε . Our main probabilistic estimate then is

Theorem 3.1. *Let $w \in \mathbb{T}$ and let $\beta < \sqrt{2}$. Then there exists $b > 0$ and $\delta_0 > 0$ such that for $\delta < \delta_0$ the Lehto integral satisfies the estimate*

$$(11) \quad \mathbb{P}(L(w, 2^{-N}, 1) < N\delta) \leq 2^{-(1+b)N}.$$

This estimate suffices to prove the existence and Hölder continuity of the solution to (6). First, only annuli centered at $w \in \partial\mathbb{D}$ need to be considered. Second, the $b > 0$ allows us to cover, for each integer N , $\partial\mathbb{D}$ by balls B_i of radii $2^{-(1+\frac{1}{2}b)N}$ such that for $\alpha > 0$

$$\text{diam}(F(B_i)) \leq C2^{-\alpha N}$$

for all i with probability $\mathcal{O}(2^{-\frac{1}{2}bN})$. A Borel-Cantelli argument then gives an a.s. Hölder continuity of F .

4. UNIQUENESS OF THE WELDING

An important issue of the welding is its uniqueness, that the curve Γ is unique up to composing with a Möbius transformation of $\widehat{\mathbb{C}}$. This would follow from the uniqueness of solutions to the Beltrami equation (6), up to a Möbius transformation. Unfortunately the control of the Lehto integrals given in Theorem 3.1 alone is much too weak to imply this. However, in our case the uniqueness of solutions to the Beltrami equation (6) is equivalent to the conformal removability of the curve $F(\mathbb{T})$. Indeed, suppose that we have two pairs f_\pm and g_\pm of solutions to eq. (1). Then the formula

$$\Psi(z) = \begin{cases} g_+ \circ (f_+)^{-1}(z) & \text{if } z \in f_+(\mathbb{D}) \\ g_- \circ (f_-)^{-1}(z) & \text{if } z \in f_-(\mathbb{D}_\infty) \end{cases}$$

defines a homeomorphism of $\widehat{\mathbb{C}}$ that is conformal outside $\Gamma = f_\pm(\mathbb{T})$. Since Γ is a Hölder curve we can invoke the result of Jones and Smirnov in [10] that Hölder curves are conformally removable i.e. that Ψ extends conformally to the entire sphere. Thus it is a Möbius transformation and uniqueness of the welding follows.

5. A LARGE DEVIATION ESTIMATE

Theorem 3.1 follows from a large deviation estimate for weakly correlated random variables. Let $\rho = 2^{-p}$ where we choose p large. Let $L_k = L_{K_f}(w, \rho^k, 2\rho^k)$ so that

$$L_{K_f}(w, 2^{-Np}, 1) \geq \sum_{k=1}^N L_k.$$

For p large L_k are Lehto integrals in well separated annuli (in logarithmic scale). Estimate (11) follows then from the inequality

$$(12) \quad \mathbb{P}\left(\sum_{k=1}^N L_k < N\delta\right) < \rho^{(1+b)N}.$$

The bound (12) is a large deviation estimate and to prove it we establish two facts: that (i) the random variables L_k are (exponentially) weakly correlated and (ii) uniformly in k , $\mathbb{P}(L_k < \varepsilon) \leq C\varepsilon$. These facts in turn rely on three ingredients: (a) an extension of ϕ to $f : \mathbb{D} \rightarrow \mathbb{D}$ with good local distortion bounds in terms of the random measure τ , (b) sharp probabilistic bounds for τ and (c) a decomposition of the free field in terms of random fields localized in scale space.

For (a) we use the classical Beurling-Ahlfors extension [6]. We pave \mathbb{D} by Whitney cubes $\{C_I\}_{I \in \mathcal{D}}$ indexed by dyadic intervals $I \subset \partial\mathbb{D}$ with $\text{diam}(C_I)$ and $\text{dist}(C_I, I)$ comparable to $|I|$. Then, extending some results by Reed on the Beurling-Ahlfors extension [13], for $z \in C_I$ we have the local distortion bound

$$(13) \quad K_f(z) \leq C \sum_{J, J'} \frac{\tau(J)}{\tau(J')}$$

where J, J' run through dyadic intervals of size $2^{-4}|I|$ lying in I and its dyadic neighbours. The virtue of this bound is that the resulting lower bound for Lehto integral L_k depends mostly on the ratios $\frac{\tau(J)}{\tau(J')}$ for J, J' dyadic intervals of size $\mathcal{O}(2^{-kp})$ and of distance $\mathcal{O}(2^{-kp})$ from w . Thus we need to understand the sizes and mutual correlations of such ratios.

For (b) we use results by Bacry and Muzy [4] and Kahane [11] on multiplicative cascades (we refer the reader to [5] for an extensive discussion of random multifractal measures). The most crucial facts are that for $\beta < \sqrt{2}$ the measure τ is non-atomic and for any interval I

$$(14) \quad \tau(I) \in L^p(\omega), \quad p \in (-\infty, 2/\beta^2).$$

Hence in particular the ratios in (13) are in $L^p(\omega)$ for $p \in [1, 2/\beta^2)$. These facts are used in the proof of statement (ii) above. The fact that we may choose $p > 1$ is crucial for our analysis and is the source for the restriction to $\beta < \sqrt{2}$.

(c) To understand the correlations between the L_k i.e. between the ratios $\frac{\tau(J)}{\tau(J')}$ on scale 2^{-kp} we use a representation due to Bacry and Muzy [4] of the free field X . It allows to decompose X as

$$(15) \quad X = \sum_{k=0}^{\infty} \zeta_k$$

where ζ_k are mutually independent a.s. continuous fields with $\zeta_k(x)$ independent from $\zeta_k(y)$ for $|x - y| > \mathcal{O}(2^{-kp})$. This decomposition leads to the following lower bound

$$(16) \quad L_n \geq m_n \exp\left(\sum_{k=0}^{n-1} 2^{-ap(n-k)} t_{n,k}\right) \left(1 + \sum_{k=n+1}^{\infty} 2^{-ap(k-n)} \ell_{n,k}\right)^{-1}$$

for $a > 0$. The main contribution here is the scale 2^{-np} contribution m_n . The positive random variables m_n are i.i.d. and satisfy the condition $\mathbb{P}(m_n < \varepsilon) \leq C\varepsilon$. Thus their sum $\sum m_n$ satisfies the estimate (12).

The corrections $t_{n,k} \geq 0$ and $\ell_{n,k} \geq 0$ represent correlations between scale 2^{-np} and scale 2^{-kp} and are multiplied with exponentially small weights in $|n - k|$. $t_{n,k}$ has gaussian tails:

$$\mathbb{P}(t_{n,k} > u) \leq ce^{-u^2/c}$$

and the $\ell_{n,k}$ has a power law tail:

$$\mathbb{P}(\ell_{n,m} > \lambda) \leq C\lambda^{-q}$$

for $q > 1$. Moreover, $t_{n,k}$ and $t_{n',k'}$ are independent if $k \neq k'$ and $\ell_{n,m}$ and $\ell_{n',m'}$ are independent if $n > m'$ or $n' > m$. These properties suffice to show that the estimate (12) extends from the m_n to the L_n .

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